Date: 2022 Dec 24

1. Circumference of a circle (Perimeter)

$$C = 2\pi R$$

2. Geometric proof:

N = sides

$$\theta = \frac{2\pi}{2N}$$

$$C_L = N\,2R\sin\theta$$

$$= \frac{N}{\pi}\,2\pi R\,\sin\frac{\pi}{N}$$

$$= 2\pi R\,\frac{\sin\left(\frac{\pi}{N}\right)}{\left(\frac{\pi}{N}\right)}$$

$$C_{upper} = N\,2R\tan\theta$$

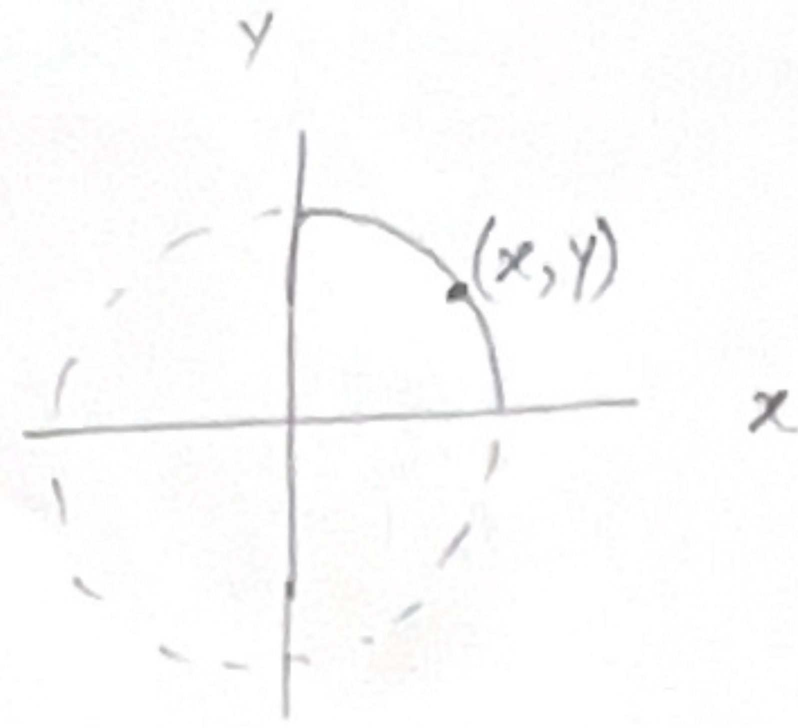
$$= 2\pi R\,\frac{\tan\left(\frac{\pi}{N}\right)}{\left(\frac{\pi}{N}\right)}$$

$$2\pi R\,\frac{\sin\left(\frac{\pi}{N}\right)}{\left(\frac{\pi}{N}\right)} < C < 2\pi R\,\frac{\tan\left(\frac{\pi}{N}\right)}{\left(\frac{\pi}{N}\right)}$$

3 Algebraic proof.

Equation: $x^2 + y^2 = R^2.$

$y = \sqrt{R^2 - x^2}$



$$C = 4 \int ds \qquad ds^2 = dx^2 + dy^2$$

$$= 4 \int_0^R dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad y = \sqrt{R^2 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{2} \frac{-2x}{\sqrt{R^2 - x^2}}$$

$$= 4 \int_0^R dx \sqrt{1 + \frac{x^2}{R^2 - x^2}} \qquad = -\frac{x}{\sqrt{R^2 - x^2}}$$

$$= 4 \int_0^R dx \sqrt{\frac{R^2}{R^2 - x^2}} \qquad x = tR, \quad dx = R\,dt.$$

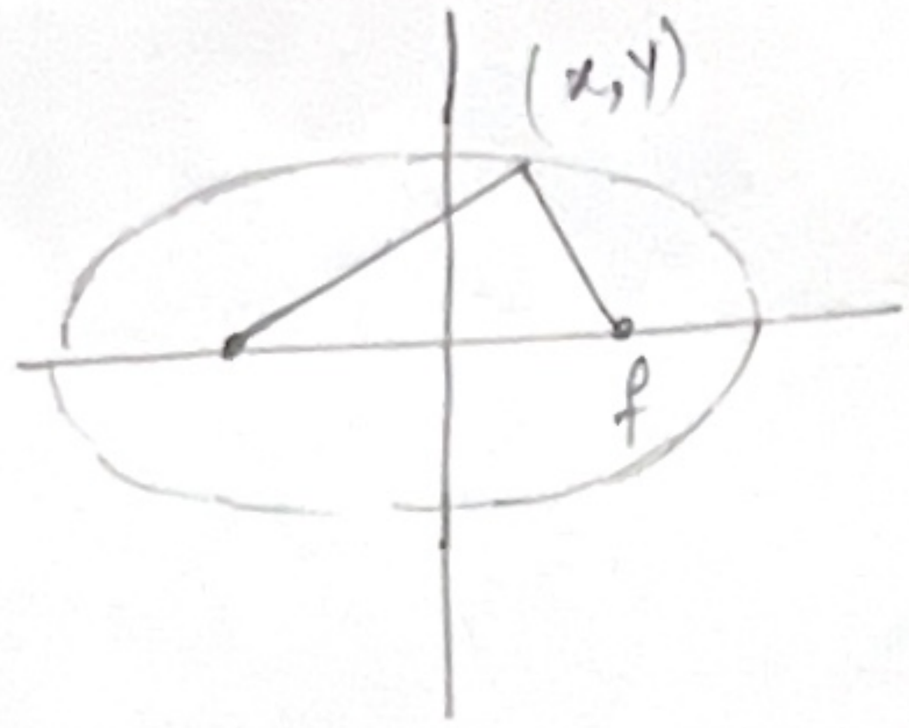
$$= 4R \int_0^1 \frac{dt}{\sqrt{1 - t^2}} \qquad t = \sin\theta, \quad dt = \cos\theta\, d\theta.$$

$$= 4R \int_0^{\frac{\pi}{2}} \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\sqrt{1 - \sin^2\theta}}}$$

$$= 4R \frac{\pi}{2}$$

$$= 2R\pi \quad ✓$$

4 Ellipse.



$$\sqrt{(x+f)^2+y^2} + \sqrt{(x-f)^2+y^2} = \text{constant}.$$

for $x=0$ let $y=b$.

$$2\sqrt{f^2+b^2} = \text{constant}.$$

for $y=0$ let $x=a$.

$$(a+f) + (a-f) = \text{constant}$$

$$\Rightarrow \quad \text{constant} = 2a.$$

and $f^2+b^2 = a^2$

$$\sqrt{(x+f)^2+y^2} + \sqrt{(x-f)^2+y^2} = 2a.$$

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$$(x+f)^2+y^2 + (x-f)^2+y^2 + 2\sqrt{}\sqrt{} = 4a^2.$$

$$2x^2+2y^2+2f^2+2\sqrt{}\sqrt{} = 4a^2.$$

$$\left[(x+f)^2+y^2\right]\left[(x-f)^2+y^2\right] = \cancel{2a^2 - x^2+y^2+f^2}$$

$$= 2a^2 - x^2 - y^2 - f^2$$

$$(x+f)^2(x-f)^2 + (x+f)^2y^2 + (x-f)^2y^2 + y^4 = (2a^2 - x^2 - y^2 - f^2)^2$$

$$x^4 + f^4 - 2x^2f^2 + x^2y^2 + f^2y^2 + 2\cancel{xf}y^2$$

$$+ \cancel{2}\, x^2y^2 + f^2y^2 - 2\cancel{xf}y^2 + y^4 = (2a^2 - x^2 - y^2 - a^2 + b^2)^2$$

$$\cancel{x^4} + f^4 - 2x^2f^2 + 2\cancel{x^2y^2} + 2y^2f^2 + \cancel{y^4}$$

$$= (x^2 + y^2 - a^2 - b^2)^2$$

$$= \cancel{x^4} + \cancel{x^2y^2} - x^2(a^2+b^2) + \cancel{x^2y^2} + \cancel{y^4} - y^2(a^2+b^2)$$

$$- x^2(a+b^2) - y^2(a^2+b^2) + (a^2+b^2)^2$$

$$(a^2-b^2)^2 - 2x^2(a^2-b^2) + 2y^2(a^2-b^2)$$

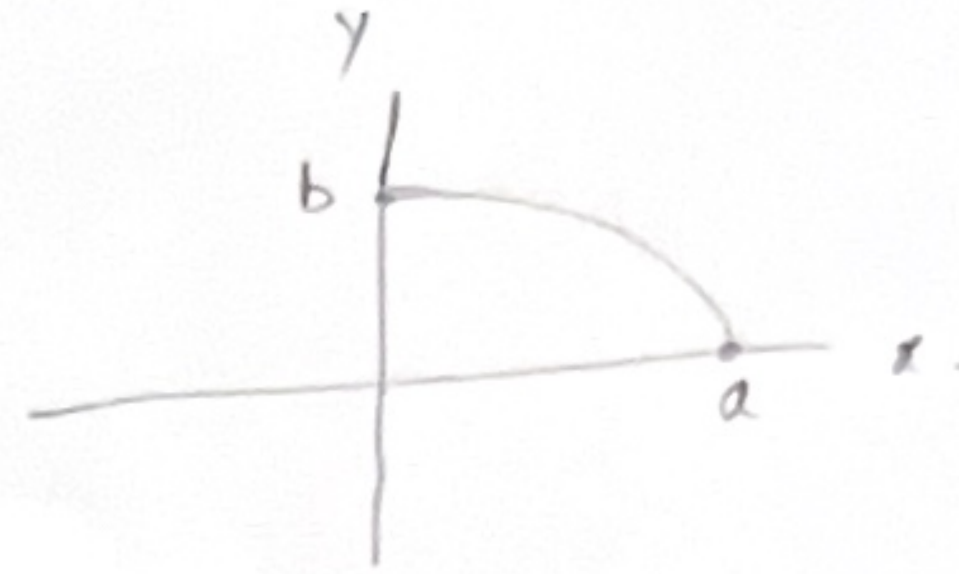
$$= -2x^2(a^2+b^2) - 2y^2(a^2+b^2) + (a^2+b^2)^2$$

$$(a^2-b^2)^2 + 4x^2b^2 + 4y^2a^2 = (a^2+b^2)^2$$

$$4x^2b^2 + 4y^2a^2 = 4a^2b^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(6) Circumference of an ellipse.



$$C = 4\int ds$$

$$= 4\int_0^a dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$$

$$= 4\int_0^a dx\sqrt{1+\frac{b^2}{a^2}\,\frac{\frac{x^2}{a^2}}{1-\frac{x^2}{a^2}}}$$

$$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\frac{dy}{dx} = b\frac{1}{\cancel{2}}\,\frac{-\cancel{2}\frac{x}{a^2}}{\sqrt{1-\frac{x^2}{a^2}}}$$

$$= -\frac{b}{a}\,\frac{\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}}$$

$$= 4a\int_0^1 dt\sqrt{\frac{1-t^2+\frac{b^2}{a^2}t^2}{1-t^2}}$$

$t = \sin\theta$

$dt = \cos\theta\, d\theta$

$$C = 4a\int_0^{\frac{\pi}{2}} \cancel{\cos\theta}\, d\theta\,\frac{\sqrt{1-\left(1-\frac{b^2}{a^2}\right)\sin^2\theta}}{\cancel{\sqrt{1-\sin^2\theta}}}$$

$$= 4a\int_0^{\frac{\pi}{2}} d\theta\sqrt{1-e^2\sin^2\theta}$$

$$e = \sqrt{1-\frac{b^2}{a^2}}$$

(7) $e=0 \Rightarrow b=a \rightarrow$ circle. $C = 4a\frac{\pi}{2} = 2a\pi$ ✓

$e=1 \Rightarrow b=0 \rightarrow$ line segment

$$C = 4a\int_0^{\pi/2} d\theta\cos\theta$$

$$= 4a\sin\theta\Big|_0^{\pi/2}$$

$$= 4a$$ ✓

(8) Complete elliptic integral of second kind

$$E(e) = \int_0^{\pi/2} d\theta \sqrt{1 - e^2 \sin^2\theta}$$

$$C = 4a\, E(e)$$

(9)
$$E(e) = \int_0^{\pi/2} d\theta \left[1 - \frac{1}{2} e^2 \sin^2\theta - \frac{1}{8} e^4 \sin^4\theta - \dots \right]$$

$$= \frac{\pi}{2} - \frac{1}{2} \cdot e^2 \frac{\pi}{4} - \frac{1}{8} e^4 \frac{3\pi}{16} - \dots$$

$$= \frac{\pi}{2} \left[1 - \frac{e^2}{4} - \frac{3}{64} e^4 - \dots \right]$$

(10)