Brachistochrone for an idealized race car on the inside surface of a cone

Osama Alkhouli,[1] Gautam Dutta,[2] Bradley Midkiff,[3,4] and K. V. Shajesh[3,5,*]

[1] *Caterpillar Inc., 100 North East Adams Street, Peoria, Illinois 61629, USA*
[2] *Dhirubhai Ambani Institute of Information and Communication Technology, Gandhinagar, Gujarat 382007, India*
[3] *Saint Edward's School, Vero Beach, Florida 32963, USA*
[4] *University of Florida, Gainesville, Florida 32611, USA*
[5] *Southern Illinois University–Carbondale, Carbondale, Illinois 62901, USA*

(Dated: January 18, 2016)

A race car, when constrained to move on the inside surface of a cone, needs to constantly maintain a speed above a minimum speed to avoid slipping inward and below a maximum threshold speed to avoid skidding outward. Calling upon an infinitely skilled driver capable of driving at the maximum threshold speed at each instance of motion, we derive exact analytic expressions for the brachistochrone for such a motion of the race car. The brachistochrone is completely determined by the geometry of the race track and is independent of the surface properties of the race track, but the time taken does depend on the surface properties. The exact analytic expression for the brachistochrone derived here is for the case when the radial component of velocity is negligible relative to the tangential component of velocity.

I. INTRODUCTION

In recent times motorized cars have become widespread, and we constantly encounter banked roads while driving on highways bending along a curve. A banked road is a road that is appropriately inclined, around a turn, to reduce the chances of vehicles skidding while maneuvering the turn. Banked roads are more striking in the case of racetracks on which the race cars move many times faster than typical cars on a highway. Nevertheless, the ubiquitous presence of banked roads all around us does not lessen the appreciation for this classic application of Newton's laws.

Here, we consider the motion of an idealized race car on the inner surface of a cone with opening angle $(\pi - 2\theta_0)$, $0 \leq \theta_0 < \pi/2$, where θ_0 is the angle the road (or racetrack) makes with the horizontal, see FIG. 1. The coordinates of a point on the surface of this road can be specified by radius r, which is the perpendicular distance of the point from the axis of the cone, and the polar angle ϕ, as described in FIG. 1. At any moment, there are primarily two forces acting on the car: the force of gravity $m\mathbf{g}$, and the force exerted by the surface on the car. The force exerted by the surface is conveniently interpreted as three independent, orthogonal, forces: the normal force $\mathbf{N}$, the force of friction in the tangential direction $\hat{\boldsymbol{\phi}}$, and the force of friction $\mathbf{f}_s$ in the direction of $\mathbf{N} \times \hat{\boldsymbol{\phi}}$, as illustrated in FIG. 2. It is useful to interpret the force exerted by the surface as a response function, which adjusts its magnitude, up to a threshold in each direction, to balance Newton's equations. Thus, part of the normal force responds, together with part of the frictional force along $\mathbf{N} \times \hat{\boldsymbol{\phi}}$, to balance the force of gravity. Similarly, part of the frictional force along $\mathbf{N} \times \hat{\boldsymbol{\phi}}$, together with part of the normal force, responds to provide the necessary radial acceleration. Further, the frictional force in the tangential direction responds to convert the torque provided by the engine into forward acceleration along the direction of $\hat{\boldsymbol{\phi}}$.

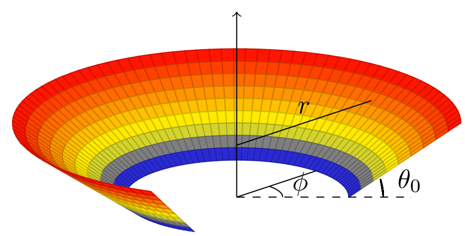

FIG. 1. A circular racetrack that is cut out of a cone of opening angle $(\pi - 2\theta_0)$. Coordinates r and ϕ define a point on the inner surface of the cone.

Let us begin by reviewing some of the well-known facts about this system. When the car is moving along a path of fixed radius r with constant speed $(dr/dt = 0, d^2\phi/dt^2 = 0)$, there exists a critical speed v_0 for which case the normal force alone completely provides the necessary centripetal force and balances the force of gravity, see FIG. 2. Thus, in this case, the frictional force along $\mathbf{N} \times \hat{\boldsymbol{\phi}}$ is completely absent, as illustrated in FIG. 2. The equations of motion for the car, in the directions of $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{g}}$ and $\hat{\mathbf{g}}$, are

$$N \sin\theta_0 = \frac{mv_0^2}{r}, \tag{1a}$$

$$N \cos\theta_0 = mg, \tag{1b}$$

respectively. The critical speed v_0 sets the scale for speed in the problem, and an explicit expression for it is obtained by dividing Eqs. (1a) and (1b),

$$v_0^2 = rg \tan\theta_0. \tag{2}$$

In particular, the physical nature of the problem changes, in the sense that the direction of the force of friction switches sign, at the critical speed v_0.

* kvshajesh@gmail.com; http://www.physics.siu.edu/~shajesh

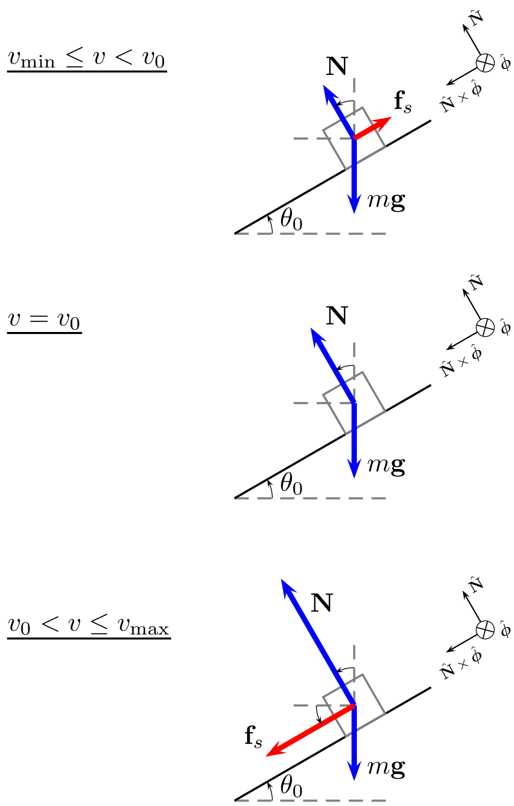

FIG. 2. Forces acting on a car moving on a banked road. The car is moving into the page in the direction of $\hat{\boldsymbol{\phi}}$. The direction of for of friction is inward (along $\hat{\mathbf{N}} \times \hat{\boldsymbol{\phi}}$) for $v_0 < v \le v_{\max}$, outward (along $-\hat{\mathbf{N}} \times \hat{\boldsymbol{\phi}}$) for $v_{\min} < v < v_0$, and the force of friction is all together absent for the critical speed $v = v_0$.

Let us begin by investigating what happens when the car deviates from this speed v_0? If the speed of the car is different from v_0, the normal force cannot balance Eqs. (1) simultaneously. Thus, as a response, the frictional force along $\mathbf{N} \times \hat{\boldsymbol{\phi}}$ gets switched on. The frictional force responds to act in the direction of $\mathbf{N} \times \hat{\boldsymbol{\phi}}$ (inwards) when the car moves faster than v_0; this provides the additional force necessary to balance the centripetal force, see FIG. 2. Similarly, the frictional force acts in the negative direction of $\mathbf{N} \times \hat{\boldsymbol{\phi}}$ (outwards) when the car moves slower than v_0, see FIG. 2. Let the frictional force in the direction of $\mathbf{N} \times \hat{\boldsymbol{\phi}}$ be represented by $\mathbf{f}_s$. Thus, for the case when the frictional force is acting inward, we have the equations of motion for the car given by,

$$N \sin\theta_0 + f_s \cos\theta_0 = \frac{mv^2}{r}, \tag{3a}$$

$$N \cos\theta_0 - f_s \sin\theta_0 = mg. \tag{3b}$$

The equations of motion for the car when the frictional force is acting outward are given by Eqs. (3) by changing the sign of f_s. Can the frictional force together with the normal force balance the centripetal force for all speeds? No. There exists an upper threshold to speed $v_{\max}$ beyond which the frictional force fails to balance the centripetal force, and it causes the car to skid outward. Similarly, there exists a lower threshold to speed $v_{\min}$ below which the car slips inward. It is convenient to define

$$f_s \le \mu_s N, \qquad \mu_s = \tan\theta_s, \tag{4}$$

where μ_s is the coefficient of static friction, and θ_s is a suitable reparametrization of the coefficient of static friction. The upper threshold for the speed is obtained by using the equality of Eq. (4) in Eq. (3) to yield

$$v_{\max}^2 = rg \tan(\theta_0 + \theta_s), \tag{5}$$

where we used the definition in Eq. (4) and the trigonometric identity for the tangent of the sum of two angles. Similarly, the lower threshold for the speed below which the car slips inward is given by

$$v_{\min}^2 = rg \tan(\theta_0 - \theta_s). \tag{6}$$

In summary, at any given point on the surface of the cone, to avoid slipping inward or skidding outward, the car has to move within speed limits described by

$$v_{\min} \le v \le v_{\max}. \tag{7}$$

A. Statement of the problem

We shall now call upon an infinitely skilled driver capable of driving at the maximum threshold speed $v_{\max}$ of Eq. (5) at every instance. Using the definition of velocity, we can write the infinitesimal statement

$$dt = \frac{ds}{v(r,\phi)}, \tag{8}$$

where dt is the infinitesimal time taken to traverse the infinitesimal displacement ds with speed $v(r,\phi)$. The time T taken to traverse an arbitrary path $r(\phi)$ on the cone of FIG. 1 by the car is then given by

$$T[r] = \int_{r(\phi)} \frac{ds}{v(r,\phi)}, \tag{9}$$

where the brackets, instead of the usual parentheses, denote the functional dependence of time T on the path. A functional is a function whose argument is a function, all possible paths in our case. For a given initial and final position of the car, specified by the points (r_i, ϕ_i) and (r_f, ϕ_f) on the cone, our goal shall be to calculate the particular path $r(\phi)$ passing through these points for which the time T is minimized.

II. VARIATIONAL PRINCIPLE

We shall now use the variational principle (of calculus) to determine the exact solution for the path that minimizes the time T. For pedagogical purpose we shall refrain from directly using the Euler-Lagrange equation for this purpose.

A. Displacement

An infinitesimal displacement in terms of cylindrical coordinates (z, r, ϕ) is

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2. \tag{10}$$

Confining the displacement to be on the surface of the cone of FIG. 1 we have the constraint $z = r \tan\theta_0$, which immediately leads to

$$ds^2 = \sec^2\theta_0\, dr^2 + r^2 d\phi^2, \tag{11}$$

using $1 + \tan^2\theta_0 = \sec^2\theta_0$. For motion on the inner surface of the cone of FIG. 1 we can define the infinitesimal displacement

$$ds = \mu(\dot{r}, r) r d\phi, \tag{12}$$

where we used the definitions

$$\mu(\dot{r}, r) = \sqrt{1 + \sec^2\theta_0 \frac{\dot{r}^2}{r^2}} \quad \text{and} \quad \dot{r} = \frac{dr}{d\phi}. \tag{13}$$

Using the definitions of Eq. (13) in Eq. (9) we have

$$T[r] = \int_{\phi_i}^{\phi_f} r d\phi\, \frac{\mu(\dot{r}, r)}{v(r, \phi)}. \tag{14}$$

The goal will be to make an infinitesimal variation in the path, δr, and use the fact that for the brachistochrone (path with least time) the corresponding variation in time, δT, will be null.

B. Variations

We start by observing that the variation in r in Eq. (14) leads to

$$T + \delta T = T[r + \delta r] = \int_{\phi_i}^{\phi_f} d\phi\, (r + \delta r) \frac{(\mu + \delta\mu)}{(v + \delta v)}, \tag{15}$$

which to the leading order presents us with the variation

$$\delta T = \int_{\phi_i}^{\phi_f} r d\phi\, \frac{\mu}{v} \left[\frac{\delta r}{r} + \frac{\delta\mu}{\mu} - \frac{\delta v}{v} \right], \tag{16}$$

where the corresponding variations in μ and v are given by

$$\delta\mu = \mu(\dot{r} + \delta\dot{r}, r + \delta r) - \mu(\dot{r}, r) = \delta\dot{r}\, \frac{\partial\mu}{\partial\dot{r}} + \delta r\, \frac{\partial\mu}{\partial r}, \tag{17a}$$

$$\delta v = v(r + \delta r, \phi) - v(r, \phi) = \delta r\, \frac{\partial v}{\partial r}. \tag{17b}$$

Thus, we have

$$\delta T = \int_{\phi_i}^{\phi_f} r d\phi\, \frac{\mu}{v} \left[\frac{\delta r}{r} + \frac{\delta\dot{r}}{\mu} \frac{\partial\mu}{\partial\dot{r}} + \frac{\delta r}{\mu} \frac{\partial\mu}{\partial r} - \frac{\delta r}{v} \frac{\partial v}{\partial r} \right]. \tag{18}$$

Using Eq. (13) we can evaluate

$$\frac{\partial\mu}{\partial r} = -\frac{\sec^2\theta_0}{\mu} \frac{\dot{r}^2}{r^3}, \quad \text{and} \quad \frac{\partial\mu}{\partial\dot{r}} = \frac{\sec^2\theta_0}{\mu} \frac{\dot{r}}{r^2}, \tag{19}$$

which when substituted in Eq. (18) leads to

$$\delta T = \int_{\phi_i}^{\phi_f} r d\phi\, \frac{\mu}{v} \left[\frac{\delta r}{r} \frac{1}{\mu^2} - \frac{\delta r}{v} \frac{\partial v}{\partial r} + \frac{\delta\dot{r}}{r} \frac{\sec^2\theta_0}{\mu^2} \frac{\dot{r}}{r} \right], \tag{20}$$

after using Eq. (13) to identify

$$1 - \frac{\sec^2\theta_0}{\mu^2} \frac{\dot{r}^2}{r^2} = \frac{1}{\mu^2}. \tag{21}$$

The term involving the variation in $\dot{r}$ can be integrated by parts to yield

$$\int_{\phi_i}^{\phi_f} r d\phi\, \frac{\mu}{v} \frac{\delta\dot{r}}{r} \frac{\sec^2\theta_0}{\mu^2} \frac{\dot{r}}{r} = \int_{\phi_i}^{\phi_f} d\phi \frac{d}{d\phi} \left(\frac{\delta r}{v} \frac{\sec^2\theta_0}{\mu} \frac{\dot{r}}{r} \right) - \int_{\phi_i}^{\phi_f} d\phi\, \delta r\, \frac{d}{d\phi} \left(\frac{1}{v} \frac{\sec^2\theta_0}{\mu} \frac{\dot{r}}{r} \right), \tag{22}$$

where we used

$$\delta\, \frac{d}{d\phi}\, r = \frac{d}{d\phi}\, \delta\, r. \tag{23}$$

The first term in Eq. (22) contributes only at the end points, and contributes zero if we consider variations such that

$$\delta r(\phi_i) = \delta r(\phi_f) = 0. \tag{24}$$

Using Eq. (22) in Eq. (20) to convert terms with $\delta\dot{r}$ into terms of δr, and requiring no variations at the end points, we have

$$\delta T = \int_{\phi_i}^{\phi_f} r d\phi\, \frac{\delta r}{r} \left[\frac{1}{\mu v} - \frac{d}{d\phi} \left(\frac{\sec^2\theta_0}{\mu v} \frac{\dot{r}}{r} \right) - \frac{r\mu}{v^2} \frac{\partial v}{\partial r} \right]. \tag{25}$$

The brachistochrone for the race car, paths of least time, are obtained by requiring the variation in the time δT in Eq. (25) to be stationary under the variation in path δr. Thus, we read out the differential equation, describing the brachistochrone, by setting the quantity inside the brackets of Eq. (25) to zero. This differential equation can be rewritten in the form

$$\frac{d}{d\phi} \left(\frac{\sec^2\theta_0}{\mu} \frac{\dot{r}}{r} \right) = \frac{1}{\mu} - \frac{1}{\mu} \frac{r}{v} \frac{\partial v}{\partial r} - \frac{1}{v} \frac{\partial v}{\partial\phi} \frac{\sec^2\theta_0}{\mu} \frac{\dot{r}}{r}, \tag{26}$$

after using

$$\frac{dv}{d\phi} = \dot{r} \frac{\partial v}{\partial r} + \frac{\partial v}{\partial\phi}. \tag{27}$$

C. Checksum: Brachistochrone on a cone

Before proceeding further, let us perform a check. Observing the fact that a cone can be mapped (or cut open) into a plane, we expect the shortest path connecting two points on a cone to be a straight line, when the particle is moving with constant speed.

For uniform speed, $\partial v/\partial r = 0$ and $\partial v/\partial \phi = 0$, Eq. (26) takes the form

$$\frac{d}{d\phi}\left(\frac{\sec^2\theta_0}{\mu}\frac{\dot r}{r}\right) = \frac{1}{\mu}. \tag{28}$$

Substituting $y = \sec\theta_0\,\dot r/r$ we can rewrite Eq. (28) as

$$\frac{dy}{1+y^2} = \cos\theta_0\,d\phi, \tag{29}$$

which can be integrated to yield the solution

$$y = \frac{1}{\cos\theta_0}\frac{1}{r}\frac{dr}{d\phi} = \tan\left(\cos\theta_0\,\phi + \phi_0\right). \tag{30}$$

This equation can be further integrated to yield, using $d(\ln\cos x) = -\tan x$,

$$r(\phi) = \frac{r_0}{\cos(\cos\theta_0\,\phi + \phi_0)}, \tag{31}$$

where r_0 and ϕ_0 are integration constants determined from initial conditions.

For initial conditions $r(\phi_i) = r_i$ and $r(\phi_f) = r_f$ we obtain the equations

$$r_0 = r_i\cos(\cos\theta_0\,\phi_i + \phi_0), \tag{32a}$$
$$r_0 = r_f\cos(\cos\theta_0\,\phi_f + \phi_0), \tag{32b}$$

which can be solved to determine r_0 and ϕ_0, and when substituted back in Eq. (31) yields the solution

$$r(\phi) = \frac{r_i r_f \sin\big(\cos\theta_0(\phi_f - \phi_i)\big)}{r_i\sin\big(\cos\theta_0(\phi-\phi_i)\big) - r_f\sin\big(\cos\theta_0(\phi-\phi_f)\big)}. \tag{33}$$

As required by the symmetry of the process, the solution remains unchanged under the interchange of the initial conditions, $(r_i,\phi_i) \leftrightarrow (r_f,\phi_f)$.

For $\theta_0 = 0$, we have a plane. Using Eq. (33) the brachistochrone on a plane, when moving with constant speed, are thus given by, setting $\cos\theta_0 = 1$,

$$r(\phi) = \frac{r_i r_f \sin(\phi_f - \phi_i)}{r_i\sin(\phi-\phi_i) - r_f\sin(\phi-\phi_f)}, \tag{34}$$

which is identified to be the equation of a straight line in cylindrical coordinates after rewriting it in the form

$$\frac{y-y_i}{x-x_i} = \frac{y_f-y_i}{x_f-x_i} \tag{35}$$

using $x = r\cos\phi$, $y = r\sin\phi$, $x_{i,f} = r_{i,f}\cos\phi_{i,f}$, and $y_{i,f} = r_{i,f}\sin\phi_{i,f}$. Thus, we verify that the path connecting two points on a plane, when moving with constant speed, is indeed a straight line.

III. BRACHISTOCHRONE OF A RACE CAR

To proceed further we need to provide the velocity profile $v(r,\phi)$ on the surface of the cone. We earlier mentioned that, to avoid slipping inward or skidding outward, the car has to continually maintain a speed above and below a threshold given by Eq. (7). We now call upon the infinitely skilled driver who can drive at the maximum threshold speed $v_{\max}$ of Eq. (5) at each instance of motion. We note that this threshold speed is independent of ϕ and its radial dependence is of the form $\sqrt{r}$. Thus, we deduce

$$\frac{r}{v}\frac{\partial v}{\partial r} = \frac{r}{\sqrt{r}}\frac{\partial\sqrt{r}}{\partial r} = \frac{1}{2}, \tag{36a}$$
$$\frac{\partial v}{\partial\phi} = 0. \tag{36b}$$

One should note that to avoid skidding the car only needs to have a tangential speed less than $v_{\max}$. We have instead required the total speed v to be less than $v_{\max}$. This is true under the approximation that the speed in the radial direction is small relative to the tangential speed. Thus, in this approximation we have

$$\frac{r}{v}\frac{\partial v}{\partial r} = \frac{r}{\sqrt{r+c(r)}}\frac{\partial\sqrt{r+c(r)}}{\partial r} = \frac{1}{2}\frac{r(1+c'(r))}{r+c(r)} \sim \frac{1}{2}. \tag{37}$$

In general, $c(r)$ is an arbitrary function of r controlled by the driver, which can not be predicted a priori. Here, $c'(r)$ denotes derivative with respect to r. We neglect $c(r)$ and set it equal to zero in the following discussion.

Starting from Eq. (26), assuming that the car moves at the maximum threshold speed without skidding, leads to the differential equation, using Eqs. (36),

$$\frac{d}{d\phi}\left(\frac{\sec^2\theta_0}{\mu}\frac{\dot r}{r}\right) = \frac{1}{2}\frac{1}{\mu}, \tag{38}$$

which is the same as the differential equation for the case of uniform speed in Eq. (28) except for a factor of 1/2. Thus, we can repeat the integrations between Eq. (28) and Eq. (31), this time with the factor of 1/2, and arrive at

$$r(\phi) = \frac{r_0}{\cos^2(\alpha\phi + \phi_0)}, \tag{39}$$

where

$$\alpha = \frac{1}{2}\cos\theta_0 \tag{40}$$

and r_0 and ϕ_0 are integration constants determined by initial conditions. We point out that the solutions in Eq. (39) are not the same as the solutions found for the brachistochrone on a cone for uniform speed. The brachistochrone for a race car given by Eq. (39)–traveling at the maximum threshold speed $v_{\max}$ of Eq. (5)-are thus obtained from the brachistochrone on a cone–at uniform speed-by introducing a factor of 2 in the power of cosine and a factor of 1/2 in the argument of ϕ in Eq. (31). In general, it is compelling to observe that, when the velocity profile is of the form $v \sim r^n$ without any angular dependence, $n \neq 1$, the brachistochrone is obtained from Eq. (39) by replacing $2 \to 1/(1-n)$ in the two occurrences. We shall not explore this generalization any further here.

For initial conditions $r(\phi_i) = r_i$ and $r(\phi_f) = r_f$, we obtain the constraints

$$\sqrt{r_f}\cos(\alpha\phi_f + \phi_0) = \pm\sqrt{r_i}\cos(\alpha\phi_i + \phi_0) \tag{41}$$

and

$$r_0 = r_i \cos^2(\alpha\phi_i + \phi_0), \tag{42}$$

which describe two minimal paths given by

$$r_\pm(\phi) = \frac{r_i r_f \sin^2\big(\alpha(\phi_f - \phi_i)\big)}{\Big[\sqrt{r_i}\sin\big(\alpha(\phi-\phi_i)\big) \pm \sqrt{r_f}\sin\big(\alpha(\phi-\phi_f)\big)\Big]^2}. \tag{43}$$

Again, as in Eq. (33), we have the symmetry under the interchange of the initial conditions, $(r_i, \phi_i) \leftrightarrow (r_f, \phi_f)$. This is stated by the relation

$$r_\pm(\phi; r_i, \phi_i; r_f, \phi_f) = r_\pm(\phi; r_f, \phi_f; r_i, \phi_i), \tag{44}$$

where we have explicitly introduced the dependence on the initial conditions. Using this notation we can write the relation between the two solutions in Eq. (43) to be

$$r_-(\phi; r_i, \phi_i; r_f, \phi_f) = r_+(\phi; r_i, \phi_i; r_f, \phi_f - \frac{\pi}{\alpha}), \tag{45}$$

which is illustrated in FIG. 3. It should be noted that the equality in Eq. (45) is in the form of the function. That is, the function $r_+(\phi; r_i, \phi_i; r_f, \phi_f - \frac{\pi}{\alpha})$ between $\phi = \phi_i$ to $\phi = \phi_f$ corresponds to the path in $r_-(\phi; r_i, \phi_i; r_f, \phi_f)$.

It is interesting to note that, the brachistochrone is determined by the geometry of the race track alone and are independent of surface properties of the race track. But, as we shall show next, the time taken to traverse these paths will depend on the surface properties of the track.

There is considerable simplification in the form of the solution in Eq. (43) for the case $r_i = r_f$:

$$r_+(\phi) \xrightarrow{r_i = r_f} r_i \frac{\sin^2\big(\frac{\alpha}{2}(\phi_f - \phi_i)\big)}{\sin^2\big(\alpha\phi - \frac{\alpha}{2}(\phi_i + \phi_f)\big)}, \tag{46a}$$

$$r_-(\phi) \xrightarrow{r_i = r_f} r_i \frac{\cos^2\big(\frac{\alpha}{2}(\phi_f - \phi_i)\big)}{\cos^2\big(\alpha\phi - \frac{\alpha}{2}(\phi_i + \phi_f)\big)}. \tag{46b}$$

The symmetry of Eq. (45) is easily verified for this case, $r_i = r_f$.

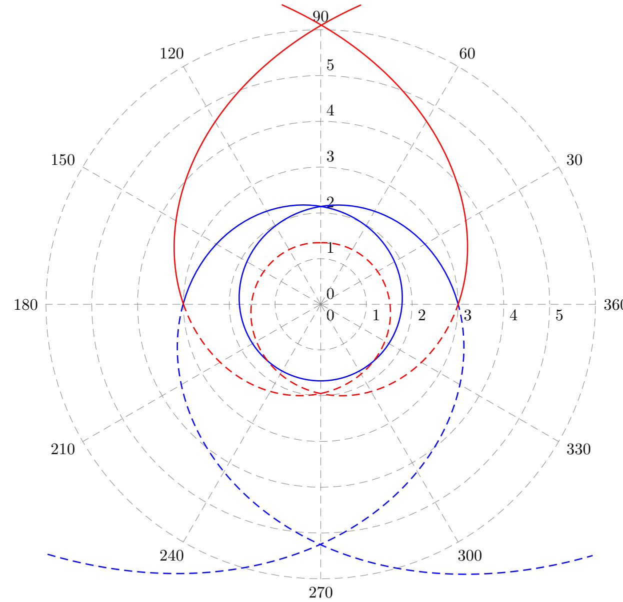

FIG. 3. Polar plot representing the two extremal paths in Eq. (43), for $\theta_0 = 72°$, $\phi_i = 0$, $\phi_f = 3\pi$, and $r_i = r_f = 3$. The bold blue curve represents the extremal path $r_-(\phi)$ in Eq. (43), and the bold red curve represents the extremal path $r_+(\phi)$ in Eq. (43) of the solution. The bold red curve starts at $\phi = 0$, wanders off to $r \to \infty$ at $\phi = 3\pi/2$, then returns back at $\phi = 3\pi$ along a symmetric path. The dotted curves extrapolate the functions beyond their initial and final points. The two solutions are related by the symmetry relation of Eq. (45).

A. Time taken for the extremal path

We shall now calculate the time taken by the car on the extremal path. We make the observation that

$$\frac{\dot r_\pm}{r_\pm} = -2\alpha \frac{\sqrt{r_i}\cos\big(\alpha(\phi-\phi_i)\big) \pm \sqrt{r_f}\cos\big(\alpha(\phi-\phi_f)\big)}{\sqrt{r_i}\sin\big(\alpha(\phi-\phi_i)\big) \pm \sqrt{r_f}\sin\big(\alpha(\phi-\phi_f)\big)}. \tag{47}$$

Using Eq. (47) in Eq. (13) to determine μ for the extremal path, and then substituting Eq. (43) in Eq. (14), we find that the time taken to traverse the extremal path involves the integral

$$\int_{\phi_i}^{\phi_f} r_\pm d\phi. \tag{48}$$

The integral in Eq. (48) can be evaluated by observing that $r_\pm(\phi)$ of Eq. (43) can be expressed as a total derivative,

$$\frac{d}{d\phi}\left[\frac{\sqrt{r_i}\cos\big(\alpha(\phi-\phi_i)\big) \pm \sqrt{r_f}\cos\big(\alpha(\phi-\phi_f)\big)}{\sqrt{r_i}\sin\big(\alpha(\phi-\phi_i)\big) \pm \sqrt{r_f}\sin\big(\alpha(\phi-\phi_f)\big)}\right]$$
$$= -\alpha \frac{r_i + r_f \pm 2\sqrt{r_i r_f}\cos\big(\alpha(\phi_f - \phi_i)\big)}{\Big[\sqrt{r_i}\sin\big(\alpha(\phi-\phi_i)\big) \pm \sqrt{r_f}\sin\big(\alpha(\phi-\phi_f)\big)\Big]^2}. \tag{49}$$

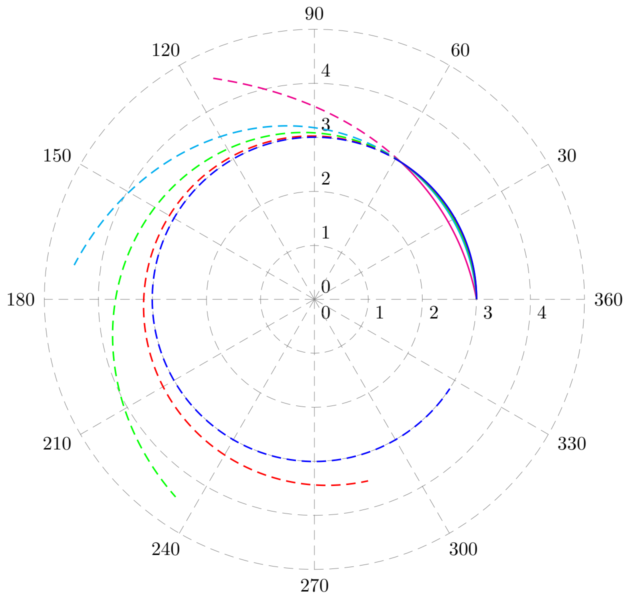

FIG. 4. The minimal path $r_-(\phi)$ of Eq. (46b), for the case when Eq. (53) is satisfied, plotted here for $\theta_0 = 30°$, $60°$, $70°$, $80°$, and $90°$. The respective $\phi_f - \phi_i = \frac{2}{\alpha}\sin^{-1}\frac{2}{\alpha}$ are 1.00798, 1.00262, 1.00122, 1.00031, and 1, respectively in radians. Here $\phi_i = 0$ and $r_i = r_f = 3$.

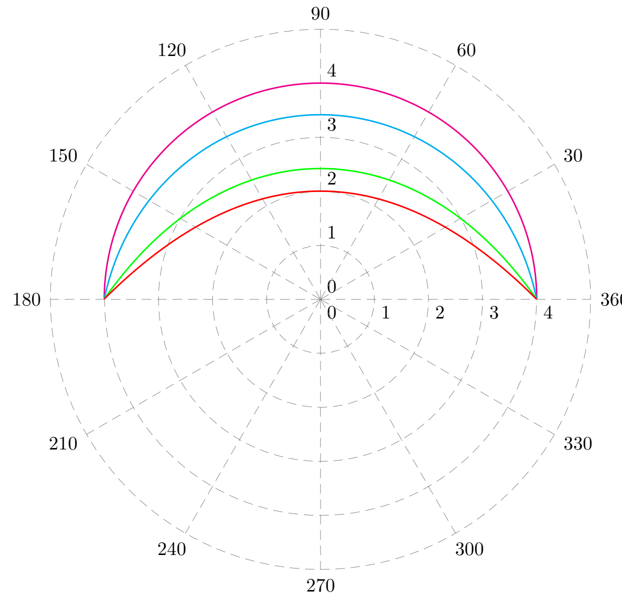

FIG. 5. Polar plot for the extremal path $r_-(\phi)$ in Eq. (43) for radius $r_f = r_i = 3$, $\phi_i = 0$, and $\phi_f = \pi$. Different paths are for varying values of θ_0. The plot for $\theta_0 = \pi/2$ is circular in this plot, but is unphysical because the opening angle of the cone in this case is zero. For the case $\theta_0 = 0$ the path passes through $r = r_i/2$ half way.

Using Eq. (49) we can immediately evaluate the integral in Eq. (48), which then leads to the time taken for the extremal paths described by Eq. (43) to be

$$T_\pm = \frac{1}{\alpha}\sqrt{\frac{r_i + r_f \pm 2\sqrt{r_i r_f}\cos\big(\alpha(\phi_f - \phi_i)\big)}{g\tan(\theta_0 + \theta_s)}}. \tag{50}$$

For the case $r_i = r_f$ we have

$$T_+ = T_0 \frac{2}{\alpha}\left|\cos\left(\frac{\alpha}{2}(\phi_f - \phi_i)\right)\right|, \tag{51a}$$

$$T_- = T_0 \frac{2}{\alpha}\left|\sin\left(\frac{\alpha}{2}(\phi_f - \phi_i)\right)\right|, \tag{51b}$$

where

$$T_0 = \sqrt{\frac{r_i}{g\tan(\theta_0 + \theta_s)}} \tag{52}$$

is the time taken to traverse a circular path of radius r_i with critical speed v_0 of Eq. (2). Since the two solutions of Eq. (43) are related by Eq. (45) both of these represent minimal paths for appropriate initial conditions.

What happens when

$$\frac{2}{\alpha}\left|\sin\left(\frac{\alpha}{2}(\phi_f - \phi_i)\right)\right| = 1? \tag{53}$$

A glance at Eq. (51) immediately informs us that $T_- = T_0$ for this case. This implies that the infinitely skilled driver does not gain on time any better than another driver who drives uniformly at critical speed v_0 of Eq. (2) along a circular path. The minimal path $r_-(\phi)$ of Eq. (46b), for the case when Eq. (53) is satisfied, takes the form

$$r_-(\phi) \sim r_i\left(1 + \frac{\alpha^2}{24}\right), \quad \text{for} \quad 0 < \phi < \phi_f = \tfrac{2}{\alpha}\sin^{-1}\tfrac{2}{\alpha}. \tag{54}$$

This is illustrated in FIG. 4, in which a few paths satisfying Eq. (53) are plotted. The specific paths hug the circle very tightly for $0 < \phi < \phi_f = \frac{2}{\alpha}\sin^{-1}\frac{2}{\alpha}$, and perfectly overlaps the circle for $\theta_0 \to \pi/2$.

IV. ANALYSIS AND CONCLUSION

The two solutions of Eq. (43) are not independent, they are related to each other by the symmetry relation in Eq. (45). Thus, it is sufficient to consider one of them for analysis, we choose $r_-(\phi)$ for this purpose. Paths corresponding to $r_-(\phi)$ for varying values of θ_0 are plotted in FIG. 5. The plots are all plotted for the case $r_i = r_f$ and $\phi_i = 0$, $\phi_f = \pi$. All the paths tend to curve inward. The car can travel faster in the outer regions of the cone, but the distance it has to cover is shorter while in the inner regions. Thus, apparently, it is not a priori obvious which part, inner or outer regions of the cone, is more economical on time. A closer analysis reveals that the distance varies as r, while speed varies as $\sqrt{r}$, thus the time taken is well estimated by

$$t = \frac{d}{v} \sim \frac{r}{\sqrt{r}} = \sqrt{r}. \tag{55}$$

Thus, traveling on the inside radius is more economical on time. The paths in FIG. 5 are minimal in the sense that they tend to spend as much time closer to the inner radius as the constraints and initial conditions allow. This is a general feature and explains the paths of all the minimal paths of a race car moving with maximum threshold speed. A few illustrative examples for the extremal path $r_-(\phi)$ of Eq. (43) are plotted in FIG. 6.

The solution presented in Eq. (43) is sufficiently rich in structure. For example, one could investigate the number of intersections on a given path. One can show that the position of the n-th intersection r_n is given by, for $r_i = r_f$,

$$r_n = r_i \frac{\cos^2\left(\frac{\alpha}{2}(\phi_f - \phi_i)\right)}{\cos^2\left(\frac{\alpha}{2} n\pi\right)}. \tag{56}$$

The number of intersections are determined by the poles of the denominator in Eq. (56). In particular, the smallest integer n that satisfies

$$\frac{1}{n+1} < \cos\theta_0 < \frac{1}{n}, \tag{57}$$

determines the number of intersections in the paths. For example, for opening angles of the cone between $60^\circ = \cos^{-1}(1/2) < \theta_0 < \cos^{-1}(1/1) = 0^\circ$ we have the number of intersections in the paths $n = 1$, $70^\circ \sim \cos^{-1}(1/3) < \theta_0 < \cos^{-1}(1/2) = 60^\circ$ we have $n = 2$, $75^\circ \sim \cos^{-1}(1/4) < \theta_0 < \cos^{-1}(1/4) \sim 70^\circ$ we have $n = 3$, and so on.

Limitations in our analysis are plenty from the perspective of direct application to real-life race car driving. Nevertheless, the derivation presented here is of academic interest, because we find exact analytic expressions for the solution to the problem posed. It is important to realize that the analysis considered here is limited to the case of a race car moving with maximum threshold speed at each instant of motion. We are further aware that the analysis carried out here is for an idealized situation because we have modeled the car as a particle. Also, real race tracks are never circular and closer to an oval shape, usually highly banked semicircular paths at the corners and straight less banked paths connecting these corners. Typical banking angles on race tracks vary between 10° to 30°, with Daytona International Speedway having the highest banking angle of 31° at the turns. Typical speeds of a race car range around 300 km/hour (∼200 miles/hour).

We have also neglected tipping. Tipping will dominate over skidding for vehicles with high center of gravity. The threshold for tipping is given by

$$v_{\max}^2 = rg\frac{b}{h}, \tag{58}$$

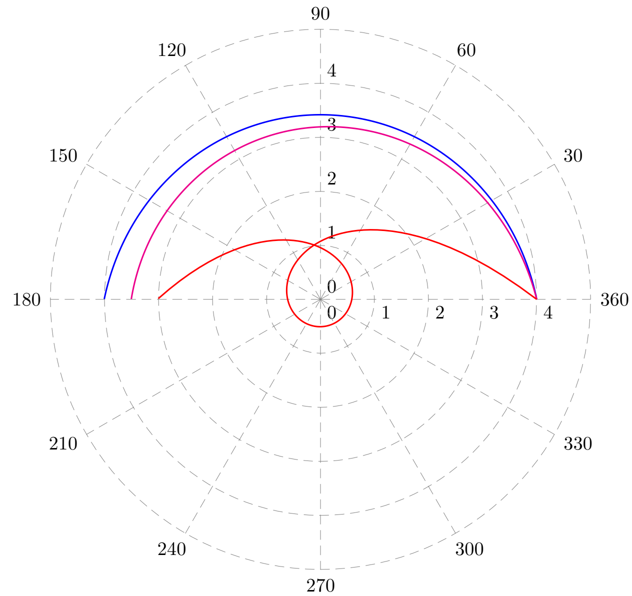

FIG. 6. Polar plot for the extremal path $r_-(\phi)$ in Eq. (43). Few illustrative examples.

where b and h are the base and height of the car respectively. But, since the radial dependence in the maximum speed for tipping is exactly the same as for sliding, we can expect our analysis covered in this article to go through with the replacement $\tan\theta_0 \to b/h$. We also neglected aerodynamics completely.

For the motion of a particle in the x-z plane, if the velocity profile is given using

$$v = \sqrt{2gz}, \tag{59}$$

z being the vertical height from a fixed point in space, the brachistochrone is the particular path connecting two points for which the particle takes the shortest time to traverse between the points. The brachistochrone on a frictionless cylinder, in Ref. [1], considers the motion of a particle constrained to move on the surface of a cone with its velocity profile given using Eq. (59). The brachistochrone for the motion of a particle on a frictionless surface of a cone has been discussed in Ref. [2], but the solution has been only reported as an integral. In our work we have considered the brachistochrone of a particle that is able to maintain the velocity profile of Eq. (5) by the energy input of the engine, which in turn leads to the solution for the brachistochrone given by Eq. (43). The brachistochrone of a rigid wheel rolling, without slipping or sliding, on the inner surface of a cone is expected to lead to similar solutions because the requirement of no slipping or sliding constraints the work done by the torque on the wheel due to the force of friction to exactly cancel the work done by the force of friction.

The exact analytic solutions derived here has been possible presumably because the cone is part of a plane when cut open. It is of interest to extend this analysis to other surfaces, say, surfaces obtained by revolution about an axis, like a hyperboloid. The work presented here should also be of interest in the studies of optimal control of autonomous vehicles.

ACKNOWLEDGMENTS

We would like to thank April Hendley for useful discussions.

[1] H. A. Yamani and A. A. Mulhem, "A cylindrical variation on the brachistochrone problem," American Journal of Physics **56**, 467 (1988).

[2] J. Gemmer, R. Umble, and M. Nolan, "Generalizations of the brachistochrone problem," arXiv:math-ph/0612052 (2006).