Tittle: Tunnelling through Earth in 42 minutes

Venue: Sphica Science Center

Date: 2022 Sep 24

(1) Gravitational force for a solid spherical mass of uniform density has the form

$$\vec{F} = \begin{cases} -\hat{r} \dfrac{GM}{R^3} r, & \text{inside}, \\ -\hat{r} \dfrac{GM}{r^2}, & \text{outside}. \end{cases}$$

(2) The associated gravitational field $\vec{g}$ satisfies

$$\vec{\nabla} \cdot \vec{g} = -4\pi G \rho_m .$$

The field lines are those of negative charges with the realization that the test charge is positive

(3) The gravitational potential V satisfies

$$\vec{g} = -\vec{\nabla} V$$

and leads to

$$V = \begin{cases} -\dfrac{3}{2} \dfrac{GM}{R} + \dfrac{1}{2} \dfrac{GM}{R^3} r^2, & \text{inside}, \\ -\dfrac{GM}{r}, & \text{outside}. \end{cases}$$

(4) Let us explicitly prove that $\vec{g} = 0$ inside a spherical shell. We use

$$\rho_m = \frac{M}{4\pi R^2} \, \delta(r' - R) .$$

Then,

$$V = - \int d^3r' \, \frac{G \rho_m}{|\vec{r} - \vec{r}'|}$$

$$= - \frac{GM}{4\pi R^2} \int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \, \frac{\delta(r' - R)}{\sqrt{r^2 + r'^2 - 2rr'\left(\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')\right)}}$$

choose r on the z-axis $(\theta = 0)$

$$= - \frac{GM}{4\pi R^2} \, 2\pi R^2 \int_0^\pi \frac{\sin\theta' d\theta'}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}}$$

$r^2 + R^2 - 2rR\cos\theta' = t$

$2rR\sin\theta' d\theta' = dt$

$$= - \frac{GM}{2} \int_{(r-R)^2}^{(r+R)^2} \frac{dt}{2rR} \frac{1}{\sqrt{t}}$$

$$= - \frac{GM}{2} \frac{1}{rR} \left[|r + R| - |r - R| \right]$$

$$= \begin{cases} - \dfrac{GM}{R} , & \text{inside} , \\ - \dfrac{GM}{r} , & \text{outside} . \end{cases}$$

$$\vec{g} = - \vec{\nabla} V = \begin{cases} 0 , & \text{inside} , \\ - \hat{r} \dfrac{GM}{r^2} , & \text{outside} . \end{cases}$$

5 For a solid sphere of uniform mass density we have

$$\rho_m = \frac{M}{\left(\frac{4\pi}{3} R^3\right)} \, \theta(R-r) .$$

$$V = - \int d^3r' \, \frac{G \rho_m}{|\vec{r} - \vec{r}'|}$$

$$= - \frac{GM}{\left(\frac{4\pi}{3} R^3\right)} \int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \, \frac{\theta(R-r')}{\sqrt{r^2 + r'^2 - 2rr'\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right)}}$$

$$= - \frac{GM}{\left(\frac{4\pi}{3} R^3\right)} \, 2\pi \int_0^R r'^2 dr' \int_0^\pi \frac{\sin\theta' d\theta'}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}}$$

$$= - \frac{GM}{R^3} \frac{3}{2} \int_0^R r'^2 dr' \, \frac{1}{rr'} \left[|r+r'| - |r-r'| \right]$$

$$= \begin{cases} - \dfrac{GM}{R^3} \dfrac{3}{2} \left[\displaystyle\int_0^r r'^2 dr' \frac{1}{rr'} 2r' + \int_r^R r'^2 dr' \frac{1}{rr'} 2r \right], & \text{inside} \\ - \dfrac{GM}{R^3} \dfrac{3}{2} \left[\displaystyle\int_0^R r'^2 dr' \frac{1}{rr'} 2r' \right], & \text{outside}, \end{cases}$$

$$= \begin{cases} - \dfrac{3}{2} \dfrac{GM}{R} + \dfrac{1}{2} \dfrac{GM}{R^3} r^2 , & \text{inside}, \\ - \dfrac{GM}{r} , & \text{outside}. \end{cases}$$

$$\vec{g} = - \vec{\nabla} V = \begin{cases} - \hat{r} \, \dfrac{GM}{R^3} r , & \text{inside}, \\ - \hat{r} \, \dfrac{GM}{r^2} , & \text{outside}. \end{cases}$$

6 Thus, inside the sphere

$$\vec{F} = -\hat{r}\left(\frac{GM}{R^3}\right) r$$

$$\left(\frac{2\pi}{T}\right)^2 = \omega_0^2$$

$(6.4)^2 = 41.0$
$(6.7)(6.0) = 40.2$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{R^3}{GM}}$$

$$= 2\pi\sqrt{\frac{(6.4\times10^6\, m)^3}{(6.7\times10^{-11})(6.0\times10^{24})}}$$

$$= 2\pi\,(8.0)(100) \text{ seconds}$$

$$\frac{T}{2} = 42 \text{ minutes}$$