Date: 2022 Aug 13

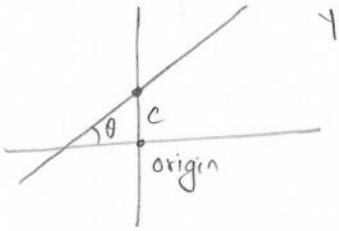
Event : Lecture Series in Theoretical Physics

Venue : Sphics Science Center, Chonadam, Thalassery (online presentation)

Title : Ford Circles

① What characterizes a straight line on a plane?

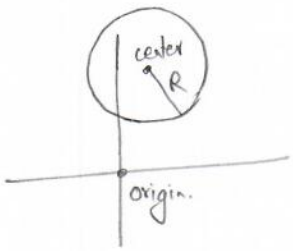
$y = mx + c$



(i) slope – quantified by θ. $(m = \tan\theta)$

(ii) distance from origin. (y intercept = c)

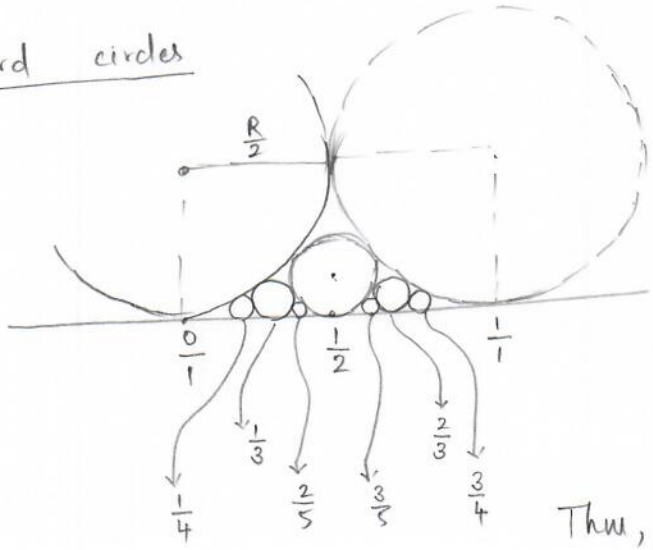
② What characterizes a circle on a plane?



(i) radius R

(ii) position of center.

③ Ford circles



(i) position of circles $\frac{m}{n} R$

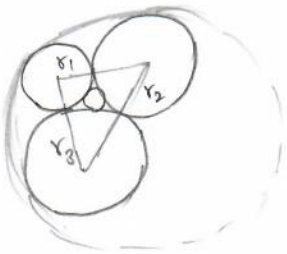
(ii) radius of circles. $\frac{R}{2n^2}$

Thus, $\left(\frac{m}{n} R, \frac{R}{2n^2}\right)$ characterizes the Ford circles.

④ Ford circles obtained using the prescription in ③ generates a circle for every rational number.

Rational number $\frac{m}{n}$ $\longleftrightarrow$ Ford circles $\left(\frac{m}{n}R, \frac{R}{2n^2}\right)$

⑤ Descartes theorem



$r_4 = ?$

Given three tangent circles, find fourth tangent to all.

curvature $c_i = \frac{1}{r_i}$

$$\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)^2 = 2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}\right)$$

This is a quadratic equation in $\frac{1}{r_4}$ and has the solution

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}$$

⑥ For Ford circles we have $r_3 \to \infty$.

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} \pm 2\frac{1}{\sqrt{r_1 r_2}}$$

$$= \left(\frac{1}{\sqrt{r_1}} \pm \frac{1}{\sqrt{r_2}}\right)^2$$

↳ we will confine to the inner circles. + root.

(7) $r_1 = \frac{1}{2}$ $r_2 = \frac{1}{2}$ $\frac{1}{r_4} = \left(\sqrt{2} + \sqrt{2}\right)^2 = 2^2 2$

$r_1 = \frac{1}{2}$ $r_2 = \frac{1}{2 2^2}$ $\frac{1}{r_4} = \left(\sqrt{2} + \sqrt{2 2^2}\right)^2 = 3^2 2$

$\vdots$

(8) Area of Ford circles

$$\left(\frac{m}{n} R, \frac{R}{2n^2}\right)$$

$$A = \pi\left(\frac{R}{2}\right)^2 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} \delta_{1,\gcd(n,m)} \pi\left(\frac{R}{2n^2}\right)^2.$$

(9)

$n=1$	$m=1$	$\phi(1) = 1$
$n=2$	$m=1,$	$\phi(2) = 1$
$n=3$	$m=1,2$	$\phi(3) = 2$
$n=4$	$m=1,3$	$\phi(4) = 2$
$n=5$	$m=1,2,3,4$	$\phi(5) = 4$
$n=6$	$m=1,5$	$\phi(6) = 2$
$n=7$	$m=1,2,3,4,5,6$	$\phi(7) = 6$
$n=8$	$m=1,3,5,7$	$\phi(8) = 4$
$n=9$	$m=1,2,4,5,7,8$	$\phi(9) = 6$
$n=10$	$m=1,3,7,9$	$\phi(10) = 4$
$n=11$	$m=1,2,3,4,5,\cdots,10$	$\phi(11) = 10$

$\vdots$

$\phi(n)$ is the Euler's totient function

(10) $$A = \frac{\pi R^2}{4} + \sum_{n=1}^{\infty} \frac{\pi R^2}{4 n^4} \sum_{m=1}^{n} \delta_{1, \gcd(n,m)}$$

$$= \frac{\pi R^2}{4} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n^4} \phi(n) \right]$$

(11) a Dirichlet series : $$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = D(s) = \frac{\zeta(s-1)}{\zeta(s)}$$

Riemann zeta function : $$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

(12) Area of Ford circles.

$$A = \frac{\pi R^2}{4} \left[1 + \frac{\zeta(3)}{\zeta(4)} \right]$$

(13) My interests :

- electromagnetic scattering off Ford circles (Berry conjecture)
- self similarity. Are Ford circles self-similar ?