Title : Curvature of the trajectory of a particle

Event : Lecture Series in Theoretical Physics.

Venue : Sphics Science Center

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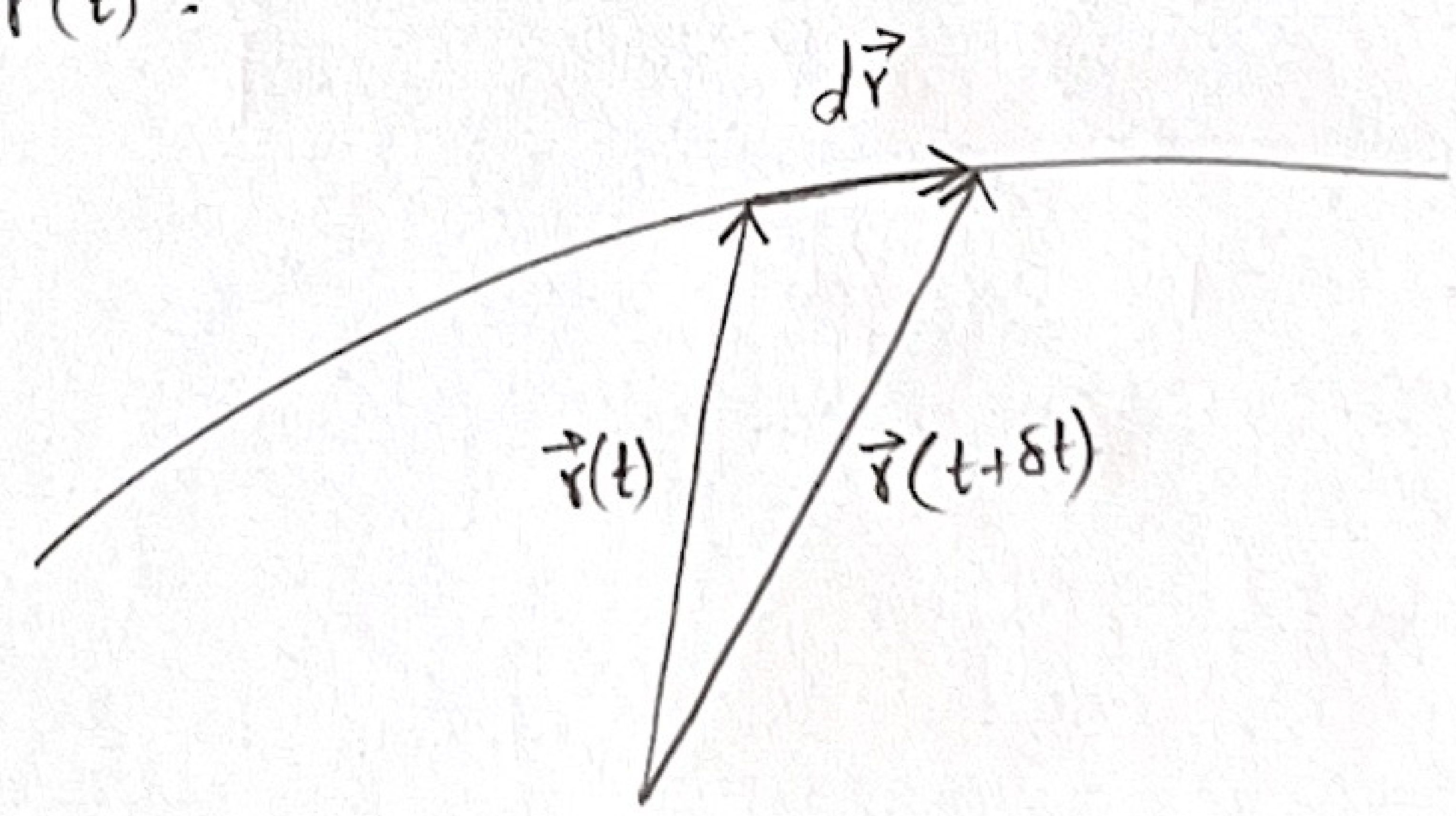
1. The trajectory of a particle is described by the evolution of the position of the particle in time. That is, it is described by $\vec{r}(t)$.



2. The instantaneous change in position is

$$d\vec{r}(t) = \vec{r}(t+\delta t) - \vec{r}(t)$$

and is in the direction of the tangent to the trajectory.

3. The velocity, which is the instantaneous change in displacement with respect to time, thus, has

4 its direction along the tangent of the trajectory. That is, velocity

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

is the rate of change of displacement.

5 It will be of interest here to separate the magnitude and direction of the velocity.

6 The magnitude of the displacement is the infinitesimal distance covered along the direction of the trajectory,

$$ds(t) = \sqrt{d\vec{r}(t) \cdot d\vec{r}(t)} = |d\vec{r}(t)|.$$

7 The speed is defined as the magnitude of velocity,

$$v(t) = |\vec{v}(t)|$$

$$= \frac{|d\vec{r}(t)|}{dt}$$

$$= \frac{ds(t)}{dt}$$

8 The distance travelled is defined as

$$s(t) - s(0) = \int_0^t dt' \; v(t')$$

$$= \int_0^t dt' \; \frac{ds(t')}{dt'}$$

$$= \int_0^t dt' \; |\vec{v}(t')| .$$

The distance travelled is a monotonous function of time. Thus, it can serve the purpose of keeping time.

9 The direction of velocity is defined as

$$\hat{v}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} .$$

10 In terms of the magnitude and direction we have

$$\vec{v}(t) = v(t) \; \hat{v}(t)$$

(11) The curvature $\vec{K}(t)$ of the trajectory is defined as the change in the direction of velocity with respect to the distance travelled,

vector

$$\vec{K}(t) = \frac{d}{ds}\hat{v}(t)$$

$$= \frac{d}{ds}\frac{\vec{v}(t)}{v(t)}$$

$$= \frac{dt}{ds}\frac{d}{dt}\frac{\vec{v}(t)}{v(t)}$$

$$= \frac{1}{v(t)}\frac{d}{dt}\frac{\vec{v}(t)}{v(t)}.$$

(12) More explicitly

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt}$$

$$\vec{K}(t) = \frac{1}{v}\frac{d}{dt}\frac{\vec{v}}{v}$$

$$= \frac{\vec{a}}{v^2} - \frac{\vec{v}}{v^3}\frac{dv}{dt}.$$

(13) We evaluate the rate of change of speed as

$$\frac{dv}{dt} = \frac{d}{dt}\sqrt{\vec{v}\cdot\vec{v}}$$

$$= \frac{1}{2}\frac{1}{\sqrt{\vec{v}\cdot\vec{v}}}\, 2\,\vec{v}\cdot\vec{a}$$

$$= \frac{\vec{v}\cdot\vec{a}}{v}$$

(14) Using (13) in (12)

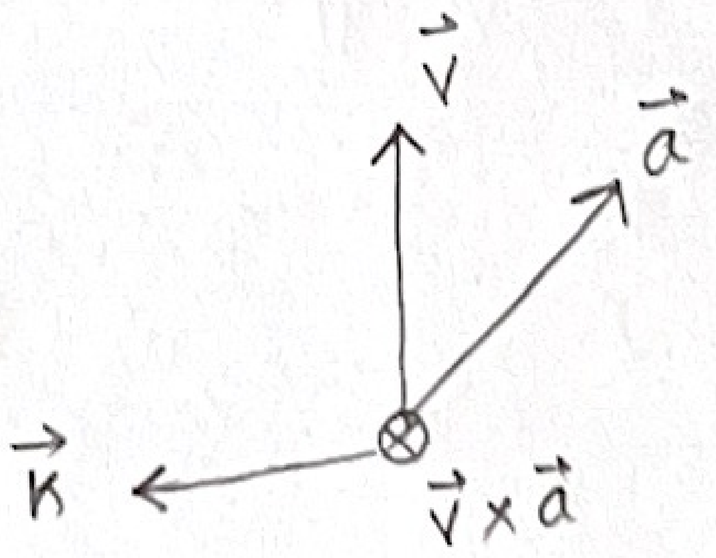
$$\vec{\kappa}(t) = \frac{\vec{a}}{v^2} - \frac{\vec{v}\,\vec{v}\cdot\vec{a}}{v^4}$$

$$= \frac{v^2\,\vec{a} - \vec{v}\,\vec{v}\cdot\vec{a}}{v^4}$$

$$= \frac{1}{v^4}\left(v^2\,\overleftrightarrow{I} - \vec{v}\,\vec{v}\right)\cdot\vec{a}$$

$$= -\frac{\vec{v}\times(\vec{v}\times\vec{a})}{v^4}$$

(15) Observe that $\vec{\kappa}$ is not necessarily in the direction of $\vec{a}$, but is always perpendicular to $\vec{v}$.



(16) The magnitude of the curvature vector is

$$\kappa(t) = |\vec{\kappa}(t)|$$

$$= \frac{|\vec{v}\times\vec{a}|}{v^3}$$

17 Example: Uniform circular motion

$\omega =$ constant, $R =$ constant

$$\vec{v} = \omega R \hat{\phi}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \omega R \frac{d\hat{\phi}}{dt} = -\omega^2 R \hat{r}$$

$$\vec{v} \times \vec{a} = \hat{z}\, \omega^3 R^2 = \hat{z} \frac{v^3}{R}$$

$$K = \frac{|\vec{v} \times \vec{a}|}{v^3} = \frac{1}{R}$$

18 Example: Uniform helical motion

$\omega =$ constant, $R =$ constant, $\dot{z} =$ constant

$$v = \sqrt{\omega^2 R^2 + \dot{z}^2}$$

$$\vec{v} = \omega R \hat{\phi} + \dot{z} \hat{z}$$

$$\vec{a} = -\omega^2 R \hat{r}$$

$$\vec{v} \times \vec{a} = \hat{z} \frac{\omega^3 R^3}{R} - \hat{\phi} \frac{\omega^2 R^2 \dot{z}}{R}$$

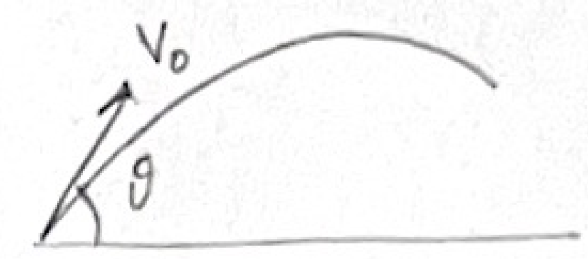
$$= \frac{\omega^2 R^2}{R} \left(\hat{z}\, \omega R - \hat{\phi}\, \dot{z} \right)$$

$$|\vec{v} \times \vec{a}| = \frac{\omega^2 R^2}{R} \sqrt{\omega^2 R^2 + \dot{z}^2}$$

$$K = \frac{|\vec{v} \times \vec{a}|}{v^3} = \frac{1}{R} \frac{\omega^2 R^2}{\omega^2 R^2 + \dot{z}^2}$$

$$= \begin{cases} \frac{1}{R}, & |\dot{z}| << \omega R, \\ 0, & \omega R << |\dot{z}|. \end{cases}$$

(19) Example: Projectile motion



$$\vec{V} = V_{0x}\,\hat{i} + (V_{0y} - gt)\,\hat{j}$$

$$V_{0x} = V_0 \cos\theta$$

$$V_{0y} = V_0 \sin\theta.$$

$$\vec{a} = 0\,\hat{i} - g\,\hat{j}$$

$$\vec{V}\times\vec{a} = -g\,V_{0x}\,\hat{k}$$

$$K = \frac{|\vec{V}\times\vec{a}|}{V^3} = \frac{g\,V_{0x}}{\left\{V_{0x}^2 + (V_{0y}-gt)^2\right\}^{3/2}}$$

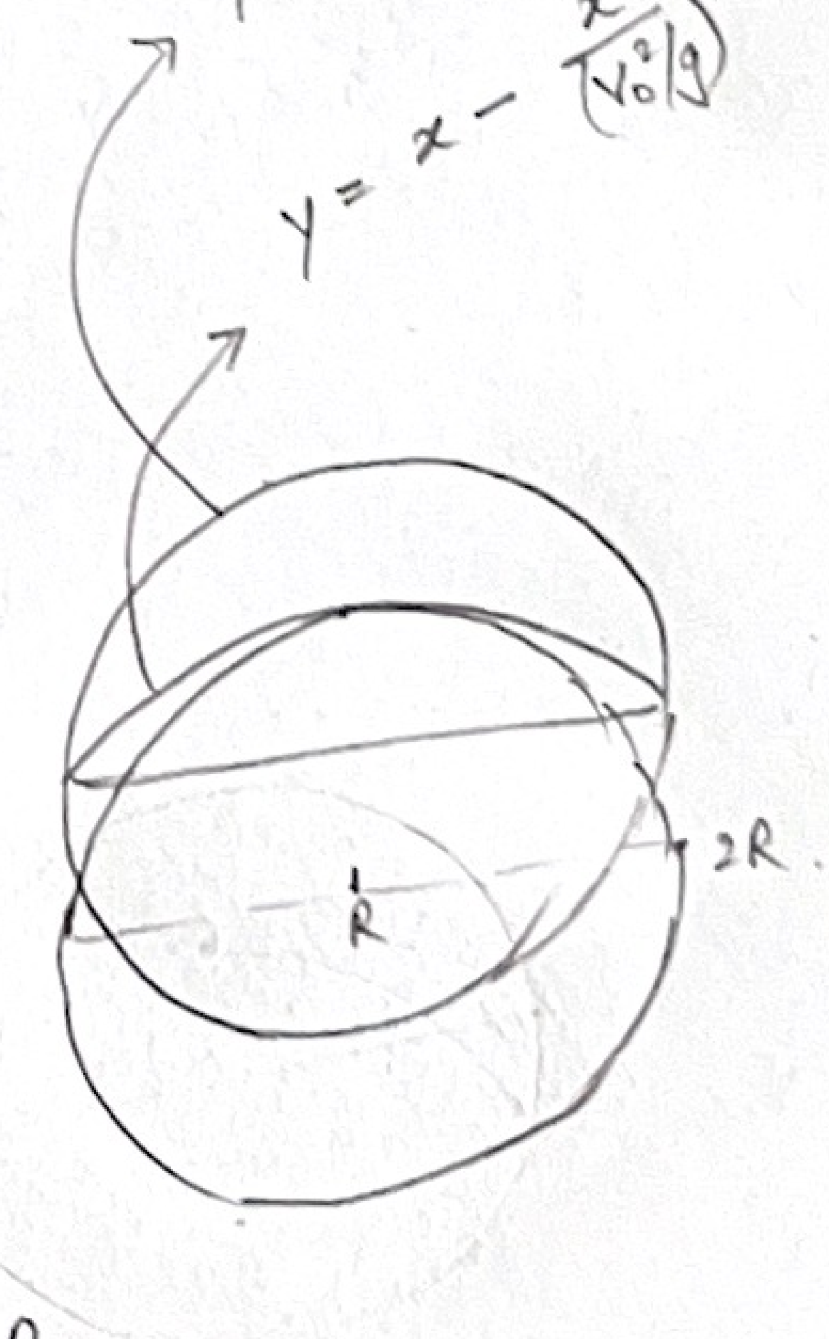
for $V_{0x} = 0$, $K = 0$.

for $V_{0x} \neq 0$, at the highest point $V_{0y} - gt = 0$, thus

$$K = \frac{g}{V_{0x}^2} = \frac{g}{V_0^2\cos^2\theta} = \frac{1}{R}.$$

$$y = \sqrt{R^2 - (x-R)^2}$$

$$y = x - \frac{x^2}{(V_0^2/g)}$$



Recall Range $D = \dfrac{V_0^2 \sin 2\theta}{g}$.

So, $\dfrac{R}{D} = \dfrac{\cos\theta}{2\sin\theta} = \dfrac{1}{2\tan\theta}$.

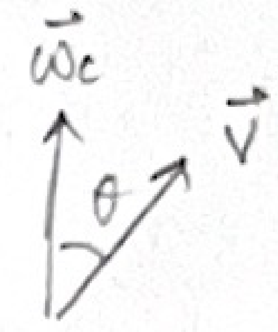
for $\theta = 45^\circ$ $\quad R = \dfrac{D}{2} = \dfrac{V_0^2}{2g}$

$$H = \frac{V_0^2\sin^2\theta}{2g} = \frac{V_0^2}{4g} = \frac{R}{2}$$

(20) Example: Charge in a magnetic field

$$m\vec{a} = q\,\vec{v}\times\vec{B} \qquad \vec{\omega}_c = \frac{q\vec{B}}{m}$$

$$\vec{a} = \vec{v}\times\vec{\omega}_c$$



$$\vec{v}\times\vec{a} = \vec{v}\times(\vec{v}\times\vec{\omega}_c)$$

$$= \vec{v}\,\vec{v}\cdot\vec{\omega}_c - v^2\vec{\omega}_c$$

$$|\vec{v}\times\vec{a}| = v\,|\vec{v}\times\vec{\omega}_c| = v^2\omega_c \sin\theta$$

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{v^3} = \frac{\omega_c}{v}\sin\theta.$$

$\nearrow \theta = 0 \quad \kappa = 0$ ✓

$\searrow \theta = \frac{\pi}{2} \quad \kappa = \frac{1}{R}$ ✓

(21) Example | Kepler problem

$$\alpha = \frac{GMm}{\mu} \qquad \frac{1}{\mu} = \frac{1}{M} + \frac{1}{m}$$

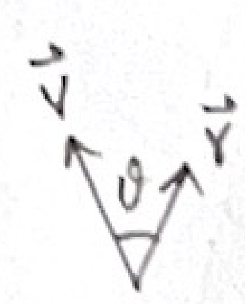
$$\mu\vec{a} = -\frac{GMm}{r^2}\hat{r}$$

$$\vec{L} = \vec{r}\times\mu\vec{v}$$

$$\vec{a} = -\alpha\frac{\vec{r}}{r^3}$$

$$\frac{d\vec{L}}{dt} = 0$$

$$\vec{v}\times\vec{a} = \alpha\frac{\vec{r}\times\vec{v}}{r^3}$$



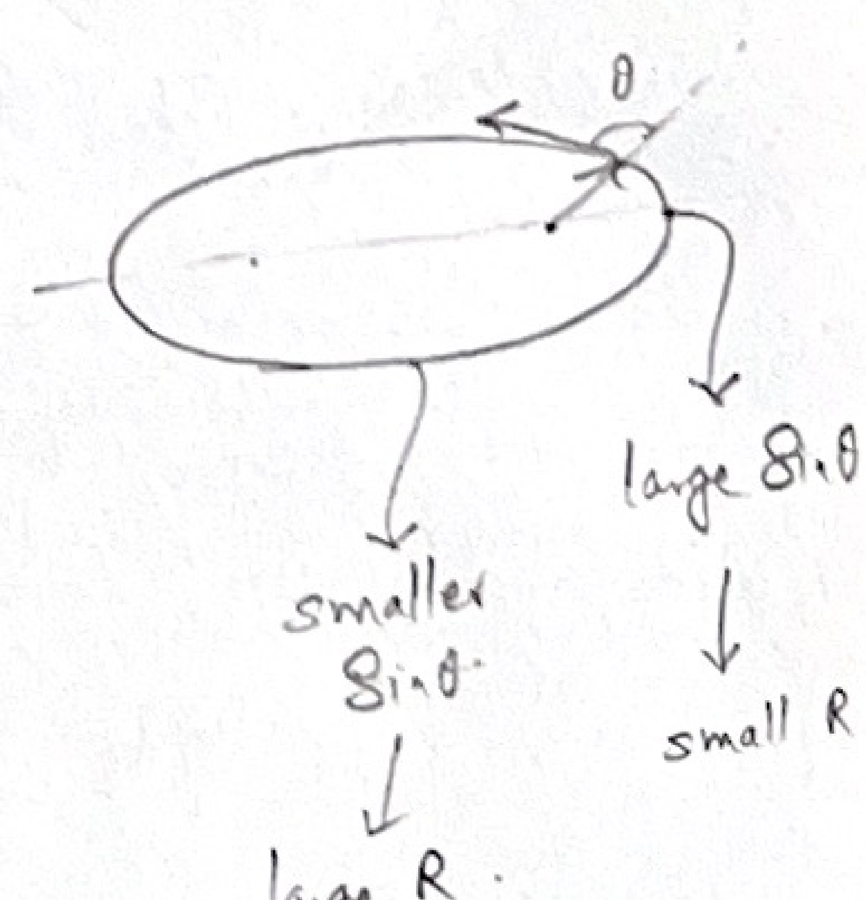
$$L = \mu v r \sin\theta$$

$$= \frac{\alpha}{\mu}\frac{\vec{L}}{r^3}$$

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{v^3} = \frac{\alpha}{\mu}\frac{L}{v^3 r^3}$$

$$= \frac{\alpha}{\mu}(\mu\sin\theta)^3\frac{L}{L^3}$$

$$= \frac{\alpha\mu^2}{L^2}\sin^3\theta.$$



(22) Example : Electric charge in the presence of a magnetic monopole.

$$m\vec{a} = \alpha\, \vec{v} \times \frac{\vec{r}}{r^3} \qquad \alpha = \frac{q_e q_m}{4\pi}$$

$$\vec{v} \times \vec{a} = \frac{\alpha}{m} \frac{\vec{v} \times (\vec{v} \times \vec{r})}{r^3}$$

$K = \frac{1}{2} m v^2$ is conserved, $\vec{L} = \vec{r} \times m\vec{v}$ is not conserved, but $|\vec{L}|$ is conserved.

The trajectory is confined on the surface of a cone with cone angle θ given by

$$\cot\theta = \frac{\alpha}{L}.$$

$$k = \frac{|\vec{v} \times \vec{a}|}{v^3} = \frac{\alpha}{m} \frac{v\, |\vec{v} \times \vec{r}|}{v^3 r^3} \qquad K = \frac{1}{2} m v^2$$

$$= \frac{\alpha}{m^2} \frac{L}{v^2 r^3}$$

$$= \frac{\alpha}{2m} \frac{L}{K} \frac{1}{r^3} \qquad r = \sqrt{(v^2 t + c)\, t + b^2}$$

↳ see EM notes.