Date: 2022 Jun 18

Event: Lecture - Series in Theoretical Physics.

Venue: Sphia Science Center

Title: Fractals: Geometric shapes with fractional demensions.

Sphia Science Center
&
Lecture-Series

→ counter-intuitive ideas
→ Science activities
→ Telescope.

1. Outline:

→ Self-similarity (or recursion)

→ Scaling and dimension

→ Fractal

→ Rayleigh scattering off fractals.

2. To illustrate a counter-intuitive feature of self-similarity we consider the series.

$$x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

recursion. →

$$= 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$

$$x = 1 + \frac{1}{2}x$$

$\Rightarrow$ $x = 2$ (intuitive)

$x = \infty$ ?

3. Next, consider the divergent series

$$x = 1 + 2 + 4 + 8 + \cdots$$

$$= 1 + 2(1 + 2 + 4 + \cdots)$$

$$x = 1 + 2x$$

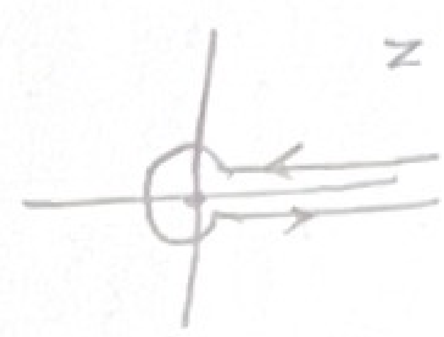
$\Rightarrow$ $x = -1$ allowed solution, regularized sum

$x = \infty$ (intuitive)

A related example

4 Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \longrightarrow \frac{\int_C \frac{dz}{z}\, z^s \frac{1}{(e^z-1)}}{\int_C \frac{dz}{z}\, z^s e^{-z}}$$




$$\zeta(-1) = 1+2+3+\dots = -\frac{1}{12}$$

5 Scaling

$$V(a) = \frac{4\pi}{3} a^3$$

$$V(2a) = 2^3\, V(a).$$

6 In general, if we have the relation

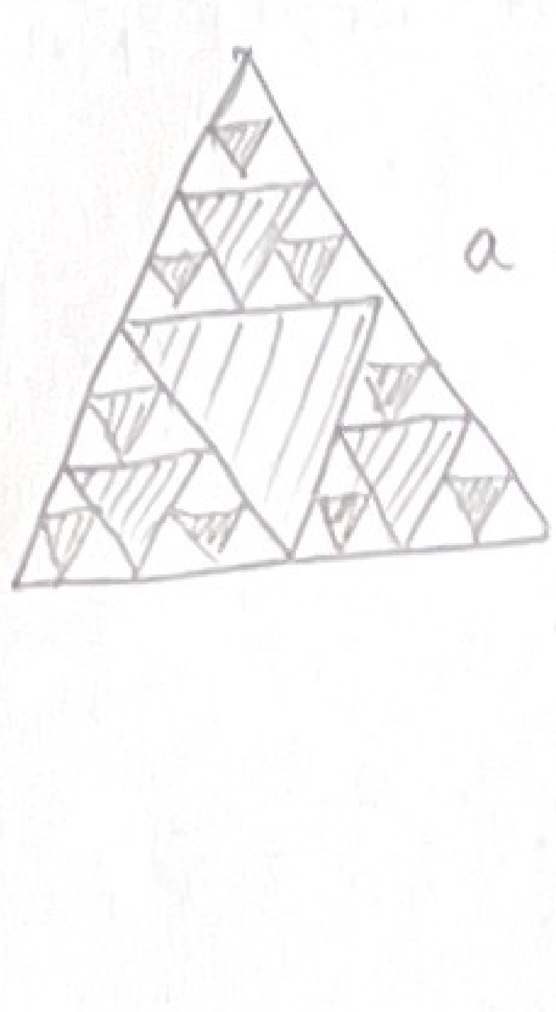
$$f(2a) = 2^\delta f(a),$$

then $\delta$ is the dimension of $f(a)$.

7 Consider a Sierpinski triangle, which is a fractal.

black – down pointing triangle – conductor

white – up – vacuum.



a

8 Area of white regions $= A_\Delta(a)$

| iteration | Area |
| --- | --- |
| 0 | $A_\Delta(a)$ |
| 1 | $3\,A_\Delta\left(\frac{a}{2}\right)$ |
| 2 | $3^2\,A_\Delta\left(\frac{a}{2^2}\right)$ |
| 3 | $3^3\,A_\Delta\left(\frac{a}{2^3}\right)$ |
| n | $3^n\,A_\Delta\left(\frac{a}{2^n}\right)$ |

$A_\Delta(a)$ = Area of an equilateral triangle of length a.

$A_S(a)$ = Area of Sierpinski triangle of length a.

$$A_S(a) = \lim_{n\to\infty} 3^n A_\Delta\left(\frac{a}{2^n}\right)$$

$$= \lim_{n\to\infty} 3^n \left(\frac{1}{2^n}\right)^2 A_\Delta(a) \quad \text{(using scaling argument)}$$

$$= \lim_{n\to\infty} \left(\frac{3}{4}\right)^n A_\Delta(a)$$

$$\to 0$$

— this result agrees with the visual intuition.

9 To obtain a counter-intuitive result, and seemingly contradictive result to 8, we use self-similarity.

$$A_S(a) = 3\,A_S\left(\frac{a}{2}\right) = 3^n A_S\left(\frac{a}{2^n}\right)$$

10 Presume that $A_s(a)$ scales as $a^\delta$. Then,

$$A_s(a) = 3 \frac{1}{2^\delta} A_s(a)$$

$$A_s(a) \left[ 1 - \frac{3}{2^\delta} \right] = 0$$

$A_s(a) = 0$ (intuitive) (OR) $1 = \frac{3}{2^\delta}$

$$\delta = \frac{\ln 3}{\ln 2} \approx 1.58$$

11 Rayleigh scattering depends on the area/volume of the scatterer and is given by

point scatterer: $$f(\theta, \phi) = -\frac{1}{\lambda^2} \tilde{\chi}(\vec{k}' - \vec{k})$$

$\vec{k} = \frac{\omega}{c} \hat{z}$ $\quad \vec{k}' = \frac{\omega}{c} \hat{r}$

$\vec{s}$ - position of scatterer

$\chi(\vec{q})$ - polarizability per unit volume of the scatterer.

$\lambda$ - wavelength of light

$$\tilde{\chi}(\vec{q}) = \int d^3r \, e^{-i\vec{q}\cdot\vec{r}} \chi(\vec{r})$$

(12) For $\theta = 0$ we have

$$\vec{k}' - \vec{k} = 0.$$

Thus,

$$f(0,0) = -\frac{1}{\lambda^2} \int d^3r \, \chi(\vec{r})$$

$$= -\frac{1}{\lambda^2} \chi \, (\text{Volume})$$

$$= -\frac{1}{\lambda^2} \sigma \, (\text{Area}).$$

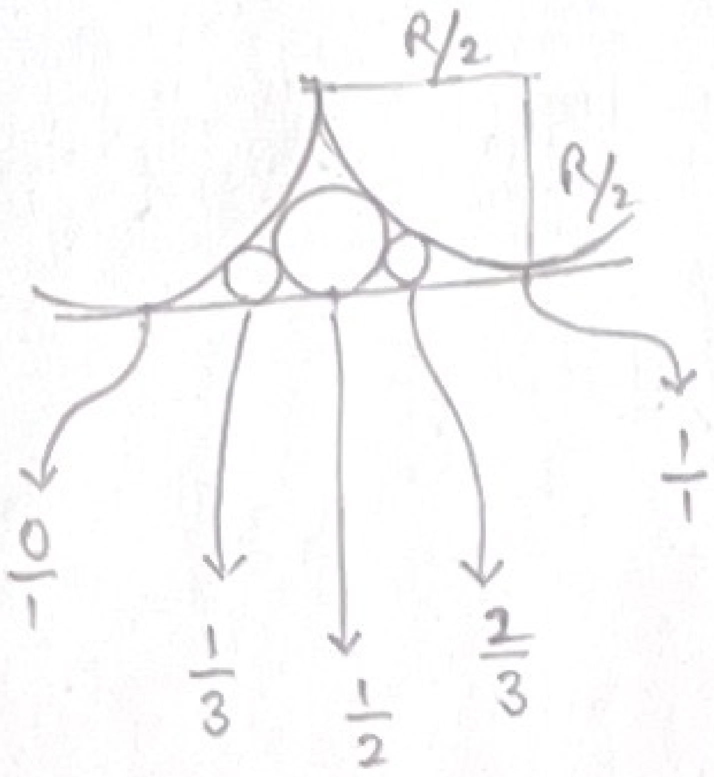
$$\sigma = \frac{\text{Polarizability}}{\text{Area}}.$$

(13) Forward scattering of Sierpinski triangle of side $a$ is

$$f(0,0) = -\frac{1}{\lambda^2} \sigma \, A_s(a)$$

$A_s(a) \longrightarrow$ area of down pointing triangles in (9).

(14) Ford circles




$$A_F = \pi \left(\frac{R}{2}\right)^2 \left[ 1 + \frac{\zeta(3)}{\zeta(4)} \right]$$